Division **S**

Year 3

MALAYSIA ASEAN SCHOOLS MATH OLYMPIADS

2013 CONTEST

SOLUTIONS AND ANSWERS

1 Let x be Tom's planned speed and y be time of travel with planned speed. The distance from A to B is xy.
Traveling at a speed 5km/h faster than planned, he will arrive 5 hours earlier:
$(x + 5)(y - 5) = xy \Leftrightarrow y - x - 5 = 0 \Leftrightarrow y = x + 5 \quad (1)$
Traveling at a speed 10km/h faster than planned, he will arrive 8 hours earlier:
$(x + 10)(y - 8) = xy \Leftrightarrow 5y - 4x - 40 = 0 \quad (2)$
Substituting (1) into (2):

$$5(x + 5) - 4x - 40 = 5x + 25 - 4x - 40 = x - 15 = 0$$
$$x = 15$$

Tom's planned speed is **15km/h**.

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2 If the first person is a noble, then the third person is a liar, which contradicts to his statement that person before him in the queue is a liar. Then the first person is the liar.
From the statement of the second person we can say that he is a noble. Again, from the statement of the third person, the third person is a liar. Continuing, the same way the following order:
Liar, Noble, Liar, Noble, Liar, Noble, ... and so on.
We can see that there are 50 ÷ 2 = **25 liars**.

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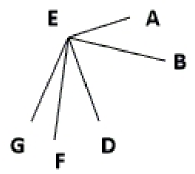
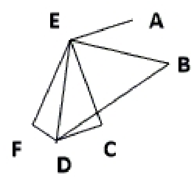
3 Let us draw a diagram. We draw a line between two letters if they played with each other.

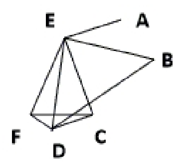
Step 1: We draw lines between F and other letters, since F has played 6 matches.



Step 2: From Step 1 we can see that player A played his only match with F. We draw the lines between D and F, C, B, since D has played 4 matches.



Step 3: From Step 2 we can see that player B has played his 2 matches with D and E. We draw lines between C and F since C has played 3 matches (2 of them with E and D).



From Step 3 we can see that all players A, B, C, D and E has played their all matches.
Therefore, F has played **3 matches**.

Items in parentheses are not required.

1	**15** (km/h)
2	**25** (Liars)
3	**3** (Matches)
4	**2** (Values)
5	**2**

4 Let the smallest angle be x. Then the other angles are $2x, 3x, \ldots, nx$
The sum of the angles of any convex n-gon is $(n-2) \times 180$
Therefore,

$$x + 2x + 3x + \cdots + nx = x(1 + 2 + 3 + \cdots + n) = x\frac{n(n+1)}{2} = 180(n-2)$$

Angle $nx < 180°$, since n-gon is convex.

$$180(n-2) = \frac{nx(n+1)}{2} < \frac{180(n+1)}{2} \quad (nx < 180°)$$

$$n - 2 < \frac{n+1}{2} \Leftrightarrow 2n - 4 < n + 1 \Leftrightarrow n < 5$$

For **2 values** of n there exists such convex n-gon.

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5 Let us write down the terms of sequence until some pattern is found:
6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, ... , (4, 2, 1)
Pattern (4, 2, 1) repeats after 6^{th} term. Every $3k^{th}$ term is 1, $(3k+1)^{st}$ term is 4, and $(3k+2)^{nd}$ term is 2 for $k \geq 3$.
Since $200 = 3 \times 66 + 2$, **the 200^{th} term is 2**.

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6 There are $6 \times 6 = 36$ total possible outcomes.
Winning outcomes: (1, 2); (2, 1); (2, 3); (3, 2); (3, 4); (4, 3); (4, 5); (5, 4); (5, 6); (6, 5) (10 winning outcomes).
Therefore, the probability that Tom wins is $\frac{10}{36} = \frac{5}{18}$.

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7 Obvious solution is when $n = 2k$ is even. We can easily check that:
$(5n)^{5n} = (5 \times 2k^2)^{5 \times 2k} = (10k)^{5k \times 2} = [(10k)^{5k}]^2$ is a perfect square. There are 100 even numbers between 1 and 200.
Next answer is when $n = 5 \times k^2$, for all positive integers k such that $1 \leq n \leq 200$.

$$(5n)^{5n} = (5 \times 5 \times k^2)^{5 \times 5 \times k^2} = (5^2 \times k^2)^{25k^2} = [(5k)^2]^{25k^2} = [(5k)^{25k^2}]^2.$$

Since we already counted all even numbers, we only need to consider when k is odd. Hence, $k = 1, 3, 5 \Rightarrow n = 5, 45, 125$.
Therefore, there are $100 + 3 =$ **103** such numbers n.

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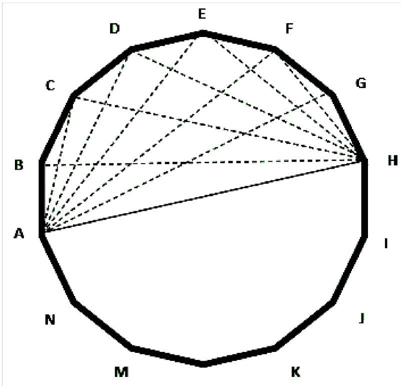
8 There are $2006 \div (3 \times 4) = 167$ multiples of 3 and 4 $(12 = 3 \times 4)$.
Also, there are
$1980 \div (3 \times 4 \times 5) = 33$ multiples of $60 = 3 \times 4 \times 5$.
Thus, the desired answer is $167 - 33 =$ **134**.

6	*Items in parentheses are not required.* $\frac{5}{18}$
7	**103**
8	**134**
9	**84**
10	**4027**

9 Any regular 14-gon is inscribed in a circle.
Let us draw a diagonal AH. We can see that it is diameter of a circle as well.



Therefore,
$\angle ABH = \angle ACH = \angle ADH = \angle AEH = \angle AFH = \angle AGH = 90^{o}$.
and triangles.
ABF, ACF, ADF, AEH, AFH and AGH are right-angled triangles. The same way, ANH, AMH, ALH, AKH, AJH, AIH are right-angled triangles.
So, for the diagonal AH, there are 12 right-angled triangles.
For each such diagonals (BI, CJ, DK, EL, FM, GN), there are 12 right-angled triangles.

Therefore, **12 × 7 = 84** right-angled triangles can be formed.

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10 $a_{n+1} = a_n + (-1)^n \times n \Rightarrow a_{n+1} - a_n = (-1)^n \times n$

$a_n - a_{n-1} = (-1)^n \times (n-1)$

$a_{n-1} - a_{n-2} = (-1)^n \times (n-2)$

…

$a_3 - a_2 = (-1)^n \times 2$

$a_2 - a_1 = (-1)^n \times 1$

Adding the equations above:

$(a_n - a_{n-1}) + \cdots + (a_2 - a_1) = a_n = (-1)^n \times 1 + (-1)^n \times 2 + \cdots + (-1)^n \times (n-1)$

$a_2 = -1$, $a_3 = 1$, $a_4 = -2$, $a_5 = 2$, $a_6 = -3$, $a_7 = 3$, $a_8 = -4$, $a_9 = 4$.

We can notice that odd terms of sequence are increasing by 1, so that $a_{2t+1} = t$ for all positive integers t.

Since $a_{2\times2013+1} = 2013$, then $a_{4027} = 2013$. The value of k is **4027**.

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11 If the first digit is 1, then the second digit is 2. Same if the first digit is 3, then the second digit is 2. The third digit will be either 1 or 3, in either case the fourth digit will be 2. Thus, we have the following representation:

$\underbrace{1 or 3}, \underbrace{2}, \underbrace{1 or 3}, \underbrace{2}, \underbrace{1 or 3}, \underbrace{2}, \underbrace{1 or 3}, \underbrace{2}, \underbrace{1 or 3}, \underbrace{2}$

There are 2 × 2 × 2 × 2 × 2 = 32 such 10-digit numbers.

If the first digit is 2, then the second digit is either 1 or 3, in either case the third digit will be 2. The fourth digit will be 1 or 3 and so on. Thus, we have the following representation:

$\underbrace{2}, \underbrace{1 or 3}, \underbrace{2}, \underbrace{1 or 3}, \underbrace{2}, \underbrace{1 or 3}, \underbrace{2}, \underbrace{1 or 3}, \underbrace{2}, \underbrace{1 or 3}$

Same way, there are 2 × 2 × 2 × 2 × 2 = 32 such 10-digit numbers.

Therefore, there are 32 + 32 = **64** such 10-digit numbers.

Division **S** Year 3	MALAYSIA ASEAN SCHOOLS MATH OLYMPIADS 2013 CONTEST	MASMO

12 For the expression to be the largest possible, b and d has to be –2 and –4 so that a^b, c^d would be positive. Let $b = -2$ and $d = -4$.
If $c = -1, -3, -5$, then $(-5)^{-4} < (-3)^{-4} < (-1)^{-4}$.
If $a = -1, -3, -5$, then $(-5)^{-2} < (-3)^{-2} < (-1)^{-2}$.

Thus, the largest possible value for the expression $a^b + c^d$ is $(-1)^{-4} + (-3)^{-2} = 1 + \frac{1}{9}$ $= \frac{10}{9}$.

13

$$\frac{1}{2\sqrt{1}+1\sqrt{2}} = \frac{1}{2\sqrt{1}+1\sqrt{2}} \times \frac{2\sqrt{1}-1\sqrt{2}}{2\sqrt{1}-1\sqrt{2}} = \frac{2\sqrt{1}-1\sqrt{2}}{\left(2\sqrt{1}\right)^2-\left(1\sqrt{2}\right)^2} = \frac{2\sqrt{1}-1\sqrt{2}}{4-2} = \frac{2\sqrt{1}}{2\times 1} - \frac{1\sqrt{2}}{2\times 1}$$

$$= \frac{\sqrt{1}}{1} - \frac{\sqrt{2}}{2}$$

$$\frac{1}{3\sqrt{2}+2\sqrt{3}} = \frac{1}{3\sqrt{2}+2\sqrt{3}} \times \frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}} = \frac{3\sqrt{2}-2\sqrt{3}}{\left(3\sqrt{2}\right)^2-\left(2\sqrt{3}\right)^2} = \frac{3\sqrt{2}-2\sqrt{3}}{3\times 2\times\left(3-1\right)}$$

$$= \frac{3\sqrt{2}}{3\times 2} - \frac{2\sqrt{3}}{3\times 2} = \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3}$$

Same way

$$\frac{1}{4\sqrt{3}+3\sqrt{4}} = \frac{\sqrt{3}}{3} - \frac{\sqrt{4}}{4}$$

…

$$\frac{1}{2013\sqrt{2012}+2012\sqrt{2013}} = \frac{\sqrt{2012}}{2012} - \frac{\sqrt{2013}}{2013}$$

Hence,

$$\frac{1}{2\sqrt{1}+1\sqrt{2}} + \frac{1}{3\sqrt{2}+2\sqrt{3}} + \frac{1}{4\sqrt{3}-3\sqrt{4}} + \cdots + \frac{1}{1000\sqrt{999}-999\sqrt{1000}}$$

$$= \left(\frac{\sqrt{1}}{1} - \frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3}\right) + \left(\frac{\sqrt{3}}{3} - \frac{\sqrt{4}}{4}\right) + \cdots \left(\frac{\sqrt{2012}}{2012} - \frac{\sqrt{2013}}{2013}\right)$$

$$= \frac{\sqrt{1}}{1} - \frac{\sqrt{2013}}{2013} = 1 - \frac{\sqrt{2013}}{2013}$$

11	*Items in parentheses are not required.* **64**
12	$\frac{10}{9}$
13	$1-\frac{\sqrt{2013}}{2013}$
14	$19\sqrt{7}$
15	**72**

Division S Year 3	MALAYSIA ASEAN SCHOOLS MATH OLYMPIADS 2013 CONTEST	

14 $\left(x+\frac{1}{x}\right)^2 = x^2+2+\frac{1}{x^2}=7$, hence $x+\frac{1}{x}=\sqrt{7}$ (1)

$25=5^2=\left(x^2+\frac{1}{x^2}\right)^2=x^4+2+\frac{1}{x^4}$, hence $x^4+\frac{1}{x^4}=25-2=23$ (2)

$5\sqrt{7}=\left(x^2+\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)=x^3+\frac{1}{x^3}+x+\frac{1}{x}=x^3+\frac{1}{x^3}+\sqrt{7}$, hence

$x^3+\frac{1}{x^3}=4\sqrt{7}$.

By multiplying (1) and (2)

$23\sqrt{7}=\left(x^4+\frac{1}{x^4}\right)\left(x+\frac{1}{x}\right)=x^5+\frac{1}{x^5}+x^3+\frac{1}{x^3}=x^5+\frac{1}{x^5}+4\sqrt{7}$, hence

$x^5+\frac{1}{x^5}=19\sqrt{7}$.

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15 $10 \times S = 5 \times 2 \times S = m^2$, hence $C = 5 \times 2 \times t^2$ for some t
$6 \times S = 2 \times 3 \times S = n^3$, hence $C = 2^2 \times 3^2 \times r^3$ for some r
$5 \times 2 \times t^2 = S = 2^2 \times 3^2 \times r^3$ or $5 \times t^2 = S = 2 \times 3^3 \times r^3$
We can see that r is divisible by 5, or r^3 is divisible by 5^3.
We can also see that t^2 is divisible 2, thus r is divisible by 2.
Hence, $r = 5 \times 2 = 10$ and $S = 2^2 \times 3^2 \times (5 \times 2)^3 = 36000$.
$36000 = 2^5 \times 3^2 \times 5^3$.
The number of positive factors is $(5 + 1) \times (2 + 1) \times (3 + 1) =$ **72**.

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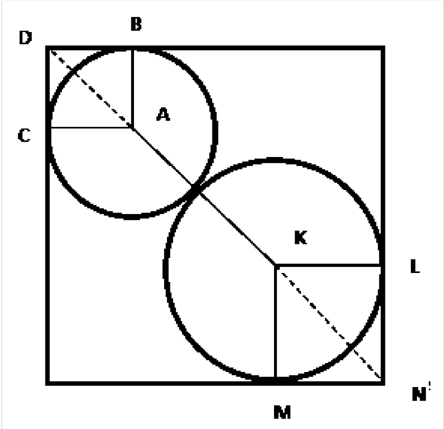
16 Let A and K be the center of circles.
Let $AB = AC = r_1$ be the radii of the smaller circle and $KL = KM = r_2$ be the radii of the bigger circle.
It is easy to notice that $ABDC$ and $KLNM$ are squares.

$DA=\sqrt{DB^2+AB^2}=\sqrt{2r_1^2}=r_1\sqrt{2}$. Same way, $KN= r_2\sqrt{2}$.

$$AK = r_1 + r_2.$$

$\sqrt{2}=DN=DA+AK+KN=r_1\sqrt{2}+r_1+r_2+r_2\sqrt{2}=(1+\sqrt{2})(r_1+r_2)$

Therefore, the sum of the lengths of the radii of the circles is $(r_1+r_2)=\frac{\sqrt{2}}{1+\sqrt{2}}=2-\sqrt{2}$.



16 *Items in parentheses are not required.*

$2-\sqrt{2}$

17

1

18

18

19

61

20

249

17 It is clear that 2^5, 12^5, 22^5, ... , 2012^5 have the same last digit. We can see that from the following example:
Find ones digit of 12^{16} and 22^{16}.

$$\begin{array}{r} 12 \\ \times 12 \\ \hline +24 \\ 12 \\ \hline 144 \end{array} \qquad \begin{array}{r} 144 \\ \times\ 12 \\ \hline +\ 288 \\ 144 \\ \hline 1728 \end{array} \qquad \begin{array}{r} 1728 \\ \times\ 12 \\ \hline +\ 3456 \\ 1728 \\ \hline 20736 \end{array}$$

Thus, the last digit of 12^{16} and 22^{16} is the same as last digit of 2^{16}.

Thus, we can say that the last digit of $1^5 + 2^5 + 3^5 + \cdots + 10^5$ is the same as the last digits of $11^5 + 12^5 + \cdots + 20^5$, $21^5 + 22^5 + \cdots + 30^5$, ... , $2001^5 + 2002^5 + \cdots + 2010^5$.

The last digit of $1^5 + 2^5 + 3^5 + \cdots + 10^5$ is 5.
Hence, the last digit of $1^5 + 2^5 + 3^5 + \cdots + 2010^5$ is 5 (201 × 5).

Since the last digit of $2011^5 + 2012^5 + 2013^5$ is 6, the last digit of $1^5 + 2^5 + 3^5 + \cdots + 2012^5 + 2013^5$ is **1** (6 + 5 = 11).

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18 The smallest 7 digit number is 1000 000. The largest is 9999 999. 9999 999 minus 1000 000 is 8999 999, but since you have a zero-based count, you have to add 1. That means there are a total of 9000 000 (or 9×10^6) different 7-digit numbers.

If you disallow sevens in all 7 digits, there are 8 ways to choose the first digit, 9 ways to choose the 2nd digit, and so on so there are $8 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 = 8 \times 9^6$ 7-digit numbers that have no sevens at all.

Hence the difference between $9 \times 10^6 - 8 \times 9^6 = 9000\,000 - 4251\,528 =$ **4748 472** is the number of 7-digit numbers with at least one seven.

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19 Let $\sqrt{2013+28\sqrt{2013+28\sqrt{\cdots}}} = A$, then
$\sqrt{2013+28A} = A$ or $2013 + 28A = A^2$ or $A^2 - 28A - 2013 = 0$
or $(A - 61)(A + 33) = 0$

Therefore $\boldsymbol{A = 61}$, since $\sqrt{2013+28\sqrt{2013+28\sqrt{2013+28\sqrt{\cdots}}}}$ is positive.

20 Multiplication by 10 = 5 × 2 gives us '0' in the end of any number. To count how many zeros does the product end, we should count the number of 5s in the product, for example, 15 = 5 × 3 has one 5 and 25 = 5 × 5 has two 5s.

From 1 to 1000, there are 1000 ÷ 5 = 200 multiples of 5, each of them has at least one 5.
25, 50, 75, 100, 150, 175, 200 have two 5s each.
From 1 to 1000, there are 1000 ÷ 25 = 40 multiples of 25, each of them has at least two 5.
120 consist of three 5s.
From 1 to 1000, there are 1000 ÷ 125 = 8 multiples of 125, each of them has at least three 5.
625 consist of four 5s.

Thus, there are 200 + 40 + 8 + 1 = 249 fives.
It is easily noticed that number of 2s is more than 5s, i.e. we will have enough of 2s to multiply by 5.
Hence, the number of zeros in the end is **249** (5 × 2 = 10).

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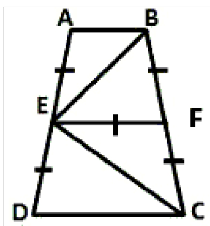
21 In any right-angled triangle median from the right angle is equal to half of the length of base side.
Therefore, $EF = BF = CF = 1$.



The length of the trapezoid median (EF) is the average length of the bases: $\frac{AB + DC}{2} = EF$.

Hence, $AB + DC = 2$.
Perimeter $= AD + BC + AB + DC = 2 + 2 + 2 =$ **6**.

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22 There are 9 one-digit palindromes: 1, 2, 3, 4, 5, 6, 7, 8, 9
There are 9 two-digit palindromes: 11, 22, 33, 44, 55, 66, 77, 88, 99
There are 10 three-digit palindromes starting with 1: 101, 111, 121, 131, 141, 151, 161, 171, 181, 191.
There are 10 three-digit palindromes starting with 2: 202, 212, 222, 232, 242, 252, 262, 272, 282, 292.
Same way, there are 70 three-digit palindromes starting with 3, 4, 5, 6, 7, 8 and 9.
Thus, there are 108 palindromes less than 1000.
There are 10 four-digit palindromes starting with 1: 1001, 1101, 1221, 1331, 1441, 1551, 1661, 1771, 1881, 1991.
There are 10 four-digit palindromes starting with 2: 2002, 2112, 2222, 2332, 2442, 2552, 2662, 2772, 2882, 2992.
Same way, there are 70 four-digit palindromes starting with 3, 4, 5, 6, 7, 8 and 9.
Thus, there are 108 + 90 = **198 palindromes** less than 10 000.

21	**61**
22	**198** *(Palindromes)*
23	**63**
24	$\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$
25	$\frac{1}{10100}$

23 Complete rectangle $ABCD$, and let CD and AB intersect in R. It is clear that R is the midpoint of PQ and that $PQ = \frac{1}{3}AB = \frac{1}{3}CD = \frac{2}{3}CR$. Let $\angle PRC = \theta$. Using the cosine rule in ΔCPR and ΔCQR respectively gives

$$PC^2 = CR^2 + PR^2 - 2CR \times PR \times \cos\theta$$

$$QC^2 = CR^2 + QR^2 - 2CR \times QR \times \cos(180° - \theta)$$

Adding, and remembering that $QR = PR$ and $\cos(180° - \theta) = -\cos\theta$, gives

$$PC^2 + QC^2 = 2CR^2 + PR^2 + QR^2$$

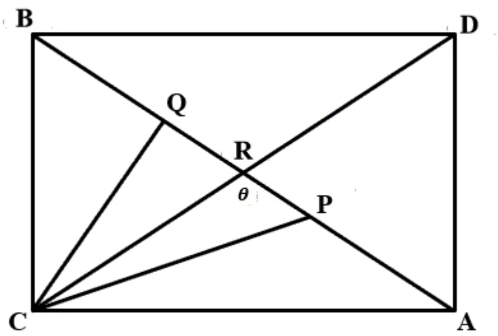
$$= 2(\frac{3}{2}PQ)^2 + 2(\frac{1}{2}PQ)^2$$

$$= 5PQ^2 = \frac{5}{9}AB^2$$

$$= \frac{5}{9}\left(CB^2 + CA^2\right)$$

$$= \frac{5}{9}\left(6^2 + 9^2\right)$$

$$= 65 = CP^2 + CQ^2$$



24 $x^4 + x^3 - 10x^2 + x + 1 = 0$

Divide the equation by $x^2 \Leftrightarrow x^2 + x - 10 + \frac{1}{x} + \frac{1}{x^2} = 0$

$$\Leftrightarrow \left(x^2 + 2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 12 = 0$$

$$\Leftrightarrow \left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 12 = 0$$

$$\Leftrightarrow \left(x + \frac{1}{x} + 4\right)\left(x + \frac{1}{x} - 3\right) = 0$$

$$\Leftrightarrow x + \frac{1}{x} + 4 = 0 \text{ or } x + \frac{1}{x} - 3 = 0$$

$$\Leftrightarrow x^2 + 4x + 1 = 0 \text{ or } x^2 - 3x + 1 = 0$$

$$\Leftrightarrow x = -2 \pm \sqrt{3} \text{ or } x = \frac{3 \pm \sqrt{5}}{2}$$

The positive solutions of the equation are $\frac{3+\sqrt{5}}{2}$, $\frac{3-\sqrt{5}}{2}$.

25 From second property we get

$$a_n\left(n^2-1\right)=a_1+a_2+\cdots+a_{n-1}$$
$$a_{n-1}\left[(n-1)^2-1\right]=a_1+a_2+\cdots+a_{n-2}$$

Combining both of above equations,

$$a_n\left(n^2-1\right)=a_{n-1}\left[(n-1)^2-1\right]+a_{n-1}=a_{n-1}(n-1)^2$$
$$\Leftrightarrow \frac{a_n}{a_{n-1}}=\frac{(n-1)^2}{n^2-1}=\frac{(n-1)^2}{(n+1)(n-1)}=\frac{n-1}{n+1}$$

Same way,

$$\frac{a_{n-1}}{a_{n-2}}=\frac{n-2}{n},\frac{a_{n-2}}{a_{n-3}}=\frac{n-3}{n-1},\cdots,\frac{a_3}{a_2}=\frac{a_2}{a_1}=\frac{1}{2}$$

Therefore,

$$\frac{a_n}{a_1}=\frac{a_n}{a_{n-1}}\times\frac{a_{n-1}}{a_{n-2}}\times\frac{a_{n-2}}{a_{n-3}}\times\cdots\times\frac{a_3}{a_2}\times\frac{a_2}{a_1}$$
$$=\frac{n-1}{n+1}\times\frac{n-2}{n}\times\frac{n-3}{n-1}\times\cdots\times\frac{2}{4}\times\frac{1}{3}$$
$$=\frac{2}{(n+1)n}$$
$$a_n=a_1\times\frac{2}{(n+1)n}=\frac{1}{(n+1)n}$$

The value of $a_{100}=\frac{1}{100\times 101}=\frac{1}{10100}$.